# Lecture 13

## 7.5 - Strategies for Integration

We have some basic forms for integrals:

| | |
|---|---|
| ① $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$ | ② $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ |
| ③ $\int e^x\,dx = e^x + C$ | ④ $\int a^x\,dx = \frac{a^x}{\ln a} + C$ |
| ⑤ $\int \sin x\,dx = -\cos x + C$ | ⑥ $\int \cos x\,dx = \sin x + C$ |
| ⑦ $\int \sec^2 x\,dx = \tan x + C$ | ⑧ $\int \csc^2 x\,dx = -\cot x + C$ |
| ⑨ $\int \sec x \tan x\,dx = \sec x + C$ | ⑩ $\int \csc x \cot x\,dx = -\csc x + C$ |
| ⑪ $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert + C$ | ⑫ $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert + C$ |
| ⑬ $\int \tan x\,dx = \ln\lvert \sec x\rvert + C$ | ⑭ $\int \cot x\,dx = \ln\lvert \sin x\rvert + C$ |
| ⑮ $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ | ⑯ $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$ $(a>0)$ |

We have a variety of techniques we can apply. If it doesn't immediately fit one of the basic forms, we use an appropriate technique:

① Use a u-sub to turn it into a basic form.

e.g. $\int e^x \sin(e^x)\,dx$ $(u=e^x)$

② Simplify the integrand/rewrite it in another way

e.g $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$, $\int \frac{\sin\theta \cot\theta}{\sec\theta}\,d\theta$

③ Apply trig integral tricks (cf. lecture 9) if relevant

e.g.: $\int \cos^n x \sin^m x\,dx$, $\int \sec^m x \tan^n x\,dx$, $\int \sin(mx)\sin(nx)\,dx$, $\int \sin(mx)\cos(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$

④ If the integrand is a rational function, try partial fraction decomposition

⑤ Look for a trig sub if there is a term of the form $\sqrt[n]{\pm x^2 \pm a^2}$ (cf. lecture 10). If there is an expression of the form $\sqrt[n]{ax+b}$, use a <u>rationalizing substitution</u>, $u=\sqrt[n]{ax+b}$. This could also work for more general things like $\sqrt[n]{g(x)}$ using $u=\sqrt[n]{g(x)}$.

⑥ Try integration by parts.

⑦ Manipulate the integrand into a more managable form

e.g.: $\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$

$\int \frac{1}{1+e^x}\,dx = \int \frac{e^x}{e^x+e^{2x}}\,dx$

could also write

$\sec x = \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

⑧ It may take multiple steps

e.g. ⓐ u-sub + partial fractions + another u-sub

ⓑ u-sub + integration by parts

ⓒ integration by parts + trig sub, etc...

Here is a list of several integrals. Outline how you would approach the integral:

(1) $\int \ln x\, dx$ $= x\ln x - \int dx = x\ln x - x + C$

IBP
$u=\ln x \quad dv=dx$
$du=\frac{1}{x}dx \quad v=x$

(2) $\int \tan x\, dx$ $= \int \frac{\sin x}{\cos x}dx = \int \frac{-1}{u}du = -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C = \ln|\sec x| + C$

rewrite $\tan x$

$u=\cos x$
$du=-\sin x\,dx$

(3) $\int \sin^3 x \cos x\, dx$ $= \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$

trig integral $\sin^m x \cos^n x$ both m&n are odd

easier to let $u=\sin x$
$du = \cos x\,dx$

(4) $\int \frac{1}{\sqrt{25-x^2}}dx$ $= \arcsin\left(\frac{x}{5}\right) + C$

(arcsin form w/ $a=5$)

Can also recognize arcsin form and do a u-sub:
$\int \frac{1}{\sqrt{25-x^2}}dx = \int \frac{1}{\sqrt{25\left(1-\left(\frac{x}{5}\right)^2\right)}}dx = \frac{1}{\cancel{5}}\int \frac{1}{\sqrt{1-u^2}}(\cancel{5}du)$

$u=\frac{x}{5}$

$= \arcsin(u) + C = \arcsin\left(\frac{x}{5}\right) + C$

(5) $\int \sec x\, dx$ $= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$

$= \int \frac{1}{u}du = \ln|u| + C = \ln|\sec x + \tan x| + C$

manipulate the integrand

$u = \sec x + \tan x$
$du = (\sec x \tan x + \sec^2 x)dx$

(6) $\int e^{\sqrt{x}}dx$ $= \int 2ue^u du = 2ue^u - \int 2e^u du = 2ue^u - 2e^u + C$

($u=\sqrt{x}$
$du = \frac{1}{2\sqrt{x}}dx \Leftrightarrow 2u\,du = dx$)

IBP
$v=2u \quad dw=e^u du$
$dv=2du \quad w=e^u$

(7) $\int \sin(7x)\cos(4x)\, dx$ $= \int \frac{1}{2}(\sin 3x + \sin 11x)dx = -\frac{1}{6}\cos 3x - \frac{1}{22}\cos 11x + C$

Use trig identity:
$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$

(8) $\int \cos^2 x\, dx$ $= \int \frac{1}{2}(1+\cos 2x)dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$

Use trig identity
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

(9) $\int \frac{1}{x^2-9}dx$ $= \int \frac{1}{(x+3)(x-3)}dx = \int \left(\frac{-1/6}{x+3} + \frac{1/6}{x-3}\right)dx = \frac{-1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + C$

$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$

(The bottom factors, so use partial fractions)

Partial fraction decomp: $\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3} \Rightarrow 1 = A(x-3) + B(x+3) = (A+B)x + (-3A+3B) \Rightarrow \begin{cases} A+B=0 \\ -3A+3B=1 \end{cases}$

Additional integrals:

(1) $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \int \frac{u}{\sqrt{1+u^2}}\,du = \int \frac{1}{2}\frac{1}{\sqrt{v}}\,dv = \frac{1}{2}\int v^{-1/2}\,dv = v^{1/2} + C$

$u = \ln x,\ du = \frac{1}{x}dx$ $\quad v = 1+u^2,\ dv = 2u\,du$

$= \sqrt{1+u^2} + C = \sqrt{1+(\ln x)^2} + C$

(2) $\int \sqrt{\frac{2-x}{2+x}}\,dx = \int \sqrt{\frac{(2-x)^2}{4-x^2}}\,dx = \int \frac{2-x}{\sqrt{4-x^2}}\,dx = \int \frac{2}{\sqrt{4-x^2}}\,dx - \int \frac{x}{\sqrt{4-x^2}}\,dx$

Turn the denominator into a difference of squares by multiplying by $\frac{2-x}{2-x}$ under the radical

arcsin form $\quad u = 4-x^2,\ du = -2x\,dx$

$= 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\int u^{-1/2}\,du = 2\arcsin\left(\frac{x}{2}\right) + \sqrt{4-x^2} + C$

(3) $\int \ln(x^2-1)\,dx = x\ln(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx = x\ln(x^2-1) - \int \left(2 + \frac{2}{x^2-1}\right)dx$

$\left(\text{IBP}: u = \ln(x^2-1)\ dv = dx,\ du = \frac{2x}{x^2-1}dx\ v = x\right)$

long division (or other method) $\quad$ partial fractions

$= x\ln(x^2-1) - \int \left(2 + \frac{-1}{x+1} + \frac{1}{x-1}\right)dx = x\ln(x^2-1) - 2x + \ln|x+1| - \ln|x-1| + C$

(4) $\int 35\arctan\sqrt{x}\,dx = \int 70u\arctan u\,du = 35u^2\arctan u - \int \frac{35u^2}{1+u^2}\,du$

$(u = \sqrt{x},\ du = \frac{1}{2\sqrt{x}}dx \Leftrightarrow 2u\,du = dx)$ $\quad \left(\text{IBP}: v = \arctan u\ \ dw = 70u\,du,\ dv = \frac{1}{1+u^2}du\ \ w = 35u^2\right)$

divide

$= 35u^2\arctan u - \int \left(35 - \frac{35}{1+u^2}\right)du = 35u^2\arctan u - 35u + 35\arctan u + C = 35\arctan(\sqrt{x})(x-1) - 35\sqrt{x} + C$

(5) $\int \frac{1+\sin x}{1-\sin x}\,dx = \int \frac{(1+\sin x)^2}{1-\sin^2 x}\,dx = \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\,dx = \int \frac{1+2\sin x+1-\cos^2 x}{\cos^2 x}\,dx$

mult by $\frac{1+\sin x}{1+\sin x}$ $\quad \sin^2 x + \cos^2 x = 1$

$= \int \frac{2+2\sin x-\cos^2 x}{\cos^2 x}\,dx = \int (2\sec^2 x + 2\sec x\tan x - 1)\,dx = 2\tan x + 2\sec x - x + C$

(6) $\int \frac{\ln x}{\sqrt{x}}\,dx = \int \ln u^2\,du = \int 2\ln u\,du = 2u\ln u - \int 2\,du$

$u = \sqrt{x},\ du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$ $\quad \text{IBP}: v = 2\ln u\ \ dw = du,\ dv = \frac{2}{u}du\ \ w = u$

$= 2u\ln u - 2u + C = 2\sqrt{x}\ln\sqrt{x} - 2\sqrt{x} + C$

$= \sqrt{x}\ln x - 2\sqrt{x} + C$

It is possible that we just cannot compute the integral. For example,

$$\int e^{x^2}\,dx, \quad \int \frac{e^x}{x}\,dx, \quad \int \frac{1}{\ln x}\,dx, \quad \int \frac{\sin x}{x}\,dx$$

cannot be computed in terms of elementary function. We can approximate them using varius methods. Another option is to use Taylor series expansions (c.f. chapter 11).